Lecture Notes for Chapter 11

International Financial Markets and Institutions

Chapter 11

The market for currency options

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Road Map

12 Summary of international finance

13 Revision classes

11.1 Main issues

- Recovering risk-neutral probabilities from option prices
- Equity options
- FX options

11.2 Recovering risk-neutral probabilities from equity index options

- Use data on put options written on the S&P500 index.
- The options will have the same maturity but different strike prices.
- Look at the way the option prices change with respect to the strike price
- Use this to estimate the risk-neutral probability distribution of the S&P500

11.2.1 Option price data

On 15th Feb 2012, you collected the following data on put options written on the S&P500 index, which mature on 30th March 2012. You may assume the time till expiry is therefore 1.5 months i.e. 1.5/12 years.

Table 11.1: S&P500 put price data

$p(t)$	0.65	0.75	1	2	3	4	4.35	5.2	8	11	14.7	21.5
K	900	950	1000	1050	1100	1125	1150	1175	1225	1250	1275	1300

On 15th Feb 2012 the S&P 500 index was at 1350.50. The continuously compounded risk - free interest rate was 2% per annum. The continuously compounded dividend yield was 2% per annum.

11.2.2 What can the data tell us about the risk-neutral probability distribution of the S&P500?

- The way option prices vary wrt the the strike price will determine the risk-neutral probability distribution of the S&P500

- Suppose the S&P500 grows exponentially with some randomness

$$S_T = S_t e^{(\mu - \frac{1}{2}\sigma^2 - q)(T-t) + \sigma(Z_T - Z_t)}$$

  where $Z_T - Z_t \sim N[0, T-t]$, $\mu - \frac{1}{2}\sigma^2$ is the exponential growth rate, q is the dividend yield, σ is the volatility of returns

- We are using physical probabilities (the $\mathbf{P}$ measure)

- The factor $e^{(\mu - \frac{1}{2}\sigma^2 - q)(T-t)}$ represents exponential growth and $e^{\sigma(Z_T - Z_0)}$ is the randomness

- The $-\frac{1}{2}\sigma^2$ is there to make sure that

$$E_t^{\mathbb{P}}[S_T e^{q(T-t)}] = S_t e^{\mu(T-t)}$$

Without it, we would have

$$E_t^{\mathbb{P}}[S_T e^{q(T-t)}] = S_t e^{(\mu+\frac{1}{2}\sigma^2)(T-t)}$$

$-\frac{1}{2}\sigma^2$ is just a correction factor, often called the Ito correction factor [Kiyoshi Ito, famous Japanese mathematician: invented Ito's formula amongst other things]

11.2.3 Returns using physical probabilities

- Expected cum-div returns

$$E_t^{\mathbb{P}}[S_T e^{q(T-t)}] = S_t e^{\mu(T-t)} \tag{11.1}$$

$$E_t^{\mathbb{P}}\left[\frac{S_T e^{q(T-t)}}{S_t}\right] = e^{\mu(T-t)} \tag{11.2}$$

$$\ln E_t^{\mathbb{P}}\left[\frac{S_T e^{q(T-t)}}{S_t}\right] = \mu(T-t) \tag{11.3}$$

11.2.4 Returns using risk-neutral probabilities

- Using risk-neutral probabilities (the $\mathbf{Q}$ measure) the expected cum-div return just depends on the risk-free rate (assume a constant risk-free rate of r per annum)

$$\ln E_t^{\mathbb{Q}}\left[\frac{S_T e^{q(T-t)}}{S_t}\right] = r(T-t)$$

- Using risk-neutral probabilities we have

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2-q)(T-t)+\sigma(Z_T-Z_t)}$$

$$\underbrace{\ln\frac{S_T}{S_t} + q(T-t)}_{\text{log cum-div return}} = \underbrace{(r-\frac{1}{2}\sigma^2)(T-t)}_{\text{non-random trend}} + \underbrace{\sigma(Z_T-Z_t)}_{\text{random noise}} \tag{11.4}$$

11.2.5 Calculating the risk-neutral probability distribution

- We want to find the risk-neutral probability of the S&P500 being below some number x at time T, conditional on the S&P500 being equal to S_t at time t.

$$\mathbb{Q}(S_T \leq x | S_t)$$

- We now use our assumption about S_T

$$S_T = S_t e^{(r - \frac{1}{2}\sigma^2 - q)(T-t) + \sigma(Z_T - Z_t)}$$

- We can show that (but we do not need the derivation for the exam)

$$\mathbb{Q}(S_T \leq x | S_t) = N(-d_2(x))$$

where

$$d_2(x) = \frac{\ln\left(\frac{S(t)}{xe^{-(r-q)(T-t)}}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}$$

and $N(x)$ is the cumulative normal distribution function

11.2.6 Connecting put prices to the risk-neutral probability distribution

- The date t Black-Scholes price of a European style put option (with strike K, expiration date T) on the S&P 500 is

$$p_t = e^{-r\,(T-t)}\,K\,N(-d_2(K)) - e^{-q(T-t)}S(t)\,N(-d_1(K)),$$

where q is the dividend yield (annualized), r is the continuously compounded (annualized) risk-free rate, $S(t)$ is the date t level of the the index. $d_1(K)$ and $d_2(K)$ are given by

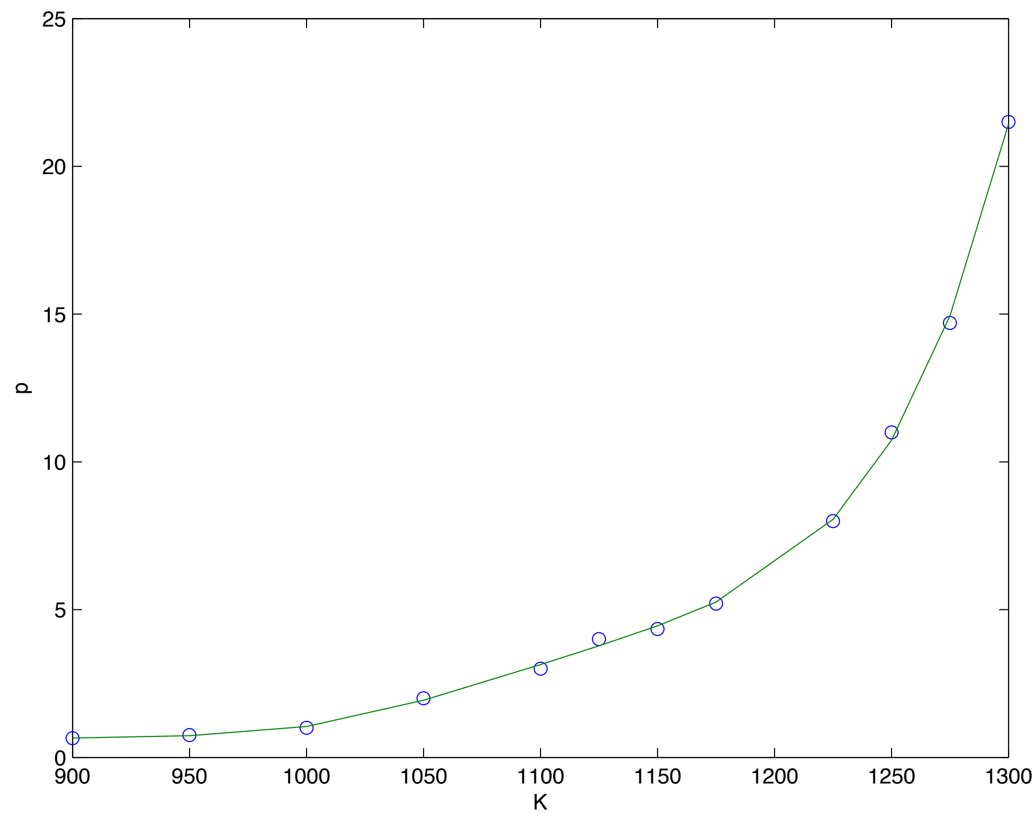
$$\begin{aligned} d_1(K) &= \frac{\ln\left(\frac{S_t}{Ke^{-(r-q)(T-t)}}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, \\ d_2(K) &= \frac{\ln\left(\frac{S_t}{Ke^{-(r-q)(T-t)}}\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. \end{aligned}$$

- We can relate the the rate of change of the put price wrt to the strike to the risk-neutral probability distribution of the S&P500

$$\frac{\partial p_t}{\partial K} = e^{-r\,(T-t)} N(-d_2(K)) = e^{-r\,(T-t)} \mathbb{Q}(S_T \leq K | S_t)$$

- If we know how put prices change wrt to strike prices, we can estimate the risk-neutral probability distribution of the S&P500

Put prices from the table (circles) together with the quintic (curved line) below.



Fitting a quintic (polynomial of degree 5) to the data: $p(K) = \sum_{n=0}^{5} a_n \left(\frac{K-1125}{B}\right)^n$

where $B = 129.7$, $a_5 = 0.6434$, $a_4 = 1.9516$, $a_3 = 1.1968$, $a_2 = 0.4584$, $a_1 = 3.3564$, $a_0 = 3.7766$.

From

$$\frac{\partial p_t}{\partial K} = e^{-r\,(T-t)}\mathbb{Q}(S_T \leq K|S_t)$$

, we have

$$\mathbb{Q}(S_T \leq K|S_t) = e^{r\,(T-t)}\frac{\partial p_t}{\partial K}$$

To find $\frac{\partial p_t}{\partial K}$, we just differentiate the quintic wrt K

11.2.7 Doing the calculation!

Option price date gives us the put price a function of the strike

$$
\begin{aligned}
p(K) &= \sum_{n=0}^{5} a_n \left(\frac{K - 1125}{B}\right)^n \\
\frac{\partial p(t)}{\partial K} &= \frac{1}{B} \sum_{n=1}^{5} n a_n \left(\frac{K - 1125}{B}\right)^{n-1}
\end{aligned}
$$

Hence

$$
\mathbb{Q}\{S_T < K | S_t\} = e^{r(T-t)} \frac{1}{B} \sum_{n=1}^{5} n a_n \left(\frac{K - 1125}{B}\right)^{n-1},
$$

For example

$$\begin{aligned}
\mathbb{Q}\{S_T < 1125 | S_t\} &= e^{r(T-t)}\frac{1}{B}\sum_{n=1}^{5} n a_n \left(\frac{1125 - 1125}{B}\right)^{n-1} \\
&= e^{r(T-t)}\frac{1}{B}a_1 \\
&= e^{0.02\times\frac{1.5}{12}}\frac{1}{129.7}\times 3.3564 \\
&= 0.025943 \\
&= 0.026\,(2\,\text{s.f.})
\end{aligned}$$

11.3 Summary

- Hidden in option pricing data is information on the risk-neutral probability distribution of the underlying

$$p_t = E_t^{\mathbb{Q}}\left[e^{-r(T-t)}\,\mathrm{max}(S_T - K, 0)\right]$$

- To extract the the risk-neutral probability distribution of the underlying, need statistical assumptions about its behavior under the risk-neutral measure

- Easy assumptions: Black-Scholes

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2-q)(T-t)+\sigma(Z_T-Z_t)},$$

where $Z_T - Z_t \sim N[0, T-t]$, $\mu - \frac{1}{2}\sigma^2$ is the exponential growth rate, q is the dividend yield, σ is the volatility of returns

- Under Black-Scholes assumptions

$$\mathbb{Q}(S_T \leq K | S_t) = e^{r\,(T-t)} \frac{\partial p_t}{\partial K}$$